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Analysis of Berry's phase by the evolution operator method

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Abstract. Berry discovered that an eigenstate undergoing an adiabatic evolution in the parameter space will acquire a topological phase. Aharonov and Anandan, on the other hand, showed that an eigenstate transporting round a closed circuit in the projective Hilbert space suffices to generate the topological phase. In this paper we shall employ the evolution operator method to study the propagation of an eigenstate. We show that Berry's phase can be represented as a closed path integral in the Hilbert space and is independent of the choice for the base $\{|n(t)\rangle\}$. The treatments of Berry, and Aharonov and Anandan, are shown to correspond to two different choices of the base. Therefore their two approaches are unified; we have acquired a more general viewpoint on the origin of Berry's phase.

1. Introduction

In 1984, Berry (1984a) discovered that any quantal system in an eigenstate transported adiabatically round a closed cycle, will acquire, apart from the usual dynamical phase, a toplogical phase known as the Berry phase. The toplogical nature of this Berry phase is elucidated by Simon (1983) (see also Niemi and Semenoff 1985) who showed that this phase is precisely the holonomy in a Hermitian line bundle. After these discoveries, a great deal of theoretical and experimental investigations have been performed. Applications have been made to different contexts of physics such as the fractional statistics of vortices in two dimensions (Haldane and Wu 1985), quantum Hall effect (Simon 1983, Arovas *et al* 1984, Semenoff and Sodano 1986) and Born-Oppenheimer approximation (Stone 1986, Moody *et al* 1986, Jackiw 1986, Martinez 1988). A classical analogue (Hannay angle) and semiclassical generalisation of Berry's phase has also been accomplished (Berry 1984b, 1985, Hannay 1985, Cina 1986, Gozzi and Thacker 1987a, b, Kugler and Shtrikman 1988, Ghosh and Dutta-Roy 1988, Anadan 1988). Besides, the gauge structure inherent in some simple quantal system is revealed (Wilczek and Zee 1984, Niemi and Semenoff 1985, Moody *et al* 1986, Jackiw 1986, Li 1987).

Apart from the above theoretical applications, experimental verification of Berry's phase has also been made to both a boson system (Chiao and Wu 1986, Tomita and Chiao 1986, Chaio *et al* 1988, Suter *et al* 1988) and a fermion system (Bitter and Dubbers 1987).

Theoretical generalisation of Berry's discoveries has been pursued in several directions. Wilczek and Zee (1984) allowed the eigenstate of the dynamical system to be degenerate and obtain non-Abelian gauge fields. Anandan and Stodolsky (1987) developed the group-theoretical method in studying Berry's phase. Aharnonov and Anandan (1987) removed the adiabatic assumption of Berry and considered Berry's phase to arise from the dynamical evolution of the quantal system round a closed circuit in the projective Hilbert space. Berry also relaxed the adiabatic assumption

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and took into account the finite rate at which a quantal system is evolving (Berry 1987). Garrison and Chiao (1988) extended the notion of Berry's phase to non-linear systems. More recently Anandan (1988) developed Aharonov and Anandan's work by using the group-theoretical method.

In the above theoretical investigations, two approaches are usually employed. One is based on the adiabatic assumption of Berry (1984a). In this approach, the toplogical phase can be expressed as a line integral in a parameter space. Eigenstates in this parameter space are also assumed to be single-valued. The other approach is due to Aharonov and Anandan (1987). The toplogical phase is now considered to result from the cyclic evolution of the quantal system itself. Superficially it seems that these two approaches come from entirely different origins. It is the purpose of this investigation to show that these two approaches are indeed two different manifestations of a more general scheme of approach. In this way we acquire a more general viewpoint on the origin of the toplogical phase in a quantal system.

In § 2, we shall give a quantum mechanical derivation of the Berry phase in a quantum system by means of the evolution operator method. In § 3, our result will be compared with those of Berry (1984a) and Anandan (1988). Section 4 concludes this paper.

2. Toplogical phase in a quantal system

We consider a quantal system described by a Hamiltonian H. This Hamiltonian can either be time dependent or time independent. Furthermore it is not necessarily cyclic in some parameter which induces the time evolution of the quantal system. Thus we allow a general scope of Hamiltonians to be considered here. The time development of the quantal system is governed by the evolution equation

$$i\hbar\frac{\partial}{\partial t}|\Phi(t)\rangle = H|\Phi(t)\rangle. \tag{2.1}$$

Introducing an evolution operator as usual by

$$|\Phi(t)\rangle = U(t,0)|\Phi(0)\rangle \tag{2.2}$$

with initial condition

$$U(0,0) = I \tag{2.3}$$

the evolution equation (2.1) will become

$$U^{+}(t,0)HU(t,0) - i\hbar U^{+}(t,0)\dot{U}(t,0) \equiv 0. \tag{2.4}$$

We would like to stress that the evolution operator is uniquely determined by the evolution equation (2.4), together with the initial condition (2.3). To proceed further, we first specify a base which is chosen to be the complete, orthonormal eigenstates of the Hamiltonian at time $t=0$ $\{|n(0)\rangle\}$

$$H(0)|n(0)\rangle = E_n|n(0)\rangle \tag{2.5}$$

where E_n is the instantaneous eigenvalue pertaining to the eigenstate $|n(0)\rangle$. Now we split the evolution operator into two unitary operators:

$$U(t,0) = \mathbb{U}(t)\mathbb{R}(t). \tag{2.6}$$

The explicit form for $\mathbb{U}(t)$, and hence $\mathbb{R}(t)$, is not specified yet. It should be noticed that there are infinitely many combinations of choices for $\mathbb{U}(t)$ and $\mathbb{R}(t)$ which, when multiplied together, will produce the unique evolution operator $U(t, 0)$. Inserting (2.6) into the identity (2.4), we obtain the following relation:

$$\mathbb{U}^{+}(t)\left[H-\mathrm{i}\hbar\frac{\partial}{\partial t}\right]\mathbb{U}(t)\equiv\mathrm{i}\hbar\dot{\mathbb{R}}(t)\mathbb{R}^{+}(t). \tag{2.7}$$

The dot appearing above $\mathbb{R}(t)$ denotes a time derivative. We denote the right-hand side of the above identity as $\mathcal{H}(t)$:

$$\mathcal{H}(t)\equiv\mathrm{i}\hbar\dot{\mathbb{R}}(t)\mathbb{R}^{+}(t). \tag{2.8}$$

$\mathcal{H}(t)$ here is evidently Hermitian.

Coming back to the evolution equation (2.1), we perform a unitary transformation to the wavefunction as follows:

$$|\Phi(t)\rangle=\mathbb{U}(t)|\Psi(t)\rangle. \tag{2.9}$$

This is essentially a change of representation from $|\Phi(t)\rangle$ to $|\Psi(t)\rangle$ so that the evolution of the quantal system in the new representation is governed by the following evolution equation:

$$\mathcal{H}(t)|\Psi(t)\rangle=\mathrm{i}\hbar\frac{\partial}{\partial t}|\Psi(t)\rangle. \tag{2.10}$$

This evolution equation shares the same form as the original evolution equation in (2.1). However this equation can readily be solved if we make a proper choice for the unitary operator $\mathbb{R}(t)$. This is done by employing the following criterion: $\mathcal{H}(t)$ in (2.8) is required to be diagonal in the base $\{|n(0)\rangle\}$. In this way, we are seeking a representation in which the associated evolution equation can be solved easily. The above requirement can be achieved if and only if $\mathbb{R}(t)$ is diagonal in the base $\{|n(0)\rangle\}$. Then the unitarity condition for $\mathbb{R}(t)$ implies that $\mathbb{R}(t)$ is given by

$$[\mathbb{R}(t)]_{m,n}=\delta_{m,n}\exp(-\mathrm{i}\vartheta_n(t)) \tag{2.11}$$

for some $\vartheta_n(t)$ satisfying the initial condition

$$\vartheta_n(0)=0. \tag{2.12}$$

Here we denote $[\mathbb{R}]_{m,n}$ as the matrix element of $\mathbb{R}$ in the base $\{|(0)\rangle\}$. From the specification made in (2.11) and (2.12), the form of the operator $\mathcal{H}(t)$ is expressible as

$$[\mathcal{H}(t)]_{m,n}=\delta_{m,n}\hbar\dot{\vartheta}_n(t). \tag{2.13}$$

Now that $\mathcal{H}(t)$ is diagonal, the evolution equation (2.10) can easily be solved to give

$$|\Psi(t)\rangle=\exp\left(-\frac{\mathrm{i}}{\hbar}\int_0^t\mathcal{H}(u)\,\mathrm{d}u\right)|\Psi(0)\rangle. \tag{2.14}$$

In the original representation $|\Phi(t)\rangle$, the solution to the evolution equation in (2.1) will then be given by

$$|\Phi(t)\rangle=\mathbb{U}(t)\exp\left(-\frac{\mathrm{i}}{\hbar}\int_0^t\mathcal{H}(u)\,\mathrm{d}u\right)|\Phi(0)\rangle. \tag{2.15}$$

This is the general expression for the wavefunction describing the evolution of the quantal system. Now suppose we start with an eigenstate of H at time $t=0$:

$$|\Phi(0)\rangle=|n(0)\rangle. \tag{2.16}$$

Then by (2.13) and (2.15), we obtain

$$|\Phi(t)\rangle = \exp(-\mathrm{i}\vartheta_n(t))\mathbb{U}(t)|n(0)\rangle. \tag{2.17}$$

In view of (2.7), (2.11) and (2.12), $\vartheta_n(t)$ is expressible as

$$\begin{aligned}\vartheta_n(t) &= \frac{1}{\hbar}\int_0^t \langle n(0)|\mathbb{U}^+(u)H\mathbb{U}(u)|n(0)\rangle\,\mathrm{d}u - \mathrm{i}\int_0^t \langle n(0)|\mathbb{U}^+(u)\dot{\mathbb{U}}(u)|n(0)\rangle\,\mathrm{d}u \\ &= \frac{1}{\hbar}\int_0^t \langle \Phi(u)|H|\Phi(u)\rangle\,\mathrm{d}u - \mathrm{i}\int_0^t \langle n(0)|\mathbb{U}^+(u)\dot{\mathbb{U}}(u)|n(0)\rangle\,\mathrm{d}u.\end{aligned} \tag{2.18}$$

Substituting (2.18) back into (2.17), the wavefunction can be rewritten in the following form:

$$|\Phi(t)\rangle = \exp(\mathrm{i}\gamma_{\mathrm{D}}(t))\exp(\mathrm{i}\gamma_{\mathrm{B}}(t))|n(t)\rangle \tag{2.19}$$

in which $\gamma_{\mathrm{D}}(t)$ is the usual dynamical phase:

$$\gamma_{\mathrm{D}}(t) = -\frac{1}{\hbar}\int_0^t \langle \Phi(u)|H|\Phi(u)\rangle\,\mathrm{d}u \tag{2.20}$$

while $\gamma_{\mathrm{B}}(t)$ is the non-dynamical phase having a similar expression as that obtained by Berry (1984a):

$$\gamma_{\mathrm{B}}(t) = \mathrm{i}\int_0^t \langle n(u)|\dot{n}(u)\rangle\,\mathrm{d}u \tag{2.21}$$

where we have denoted

$$|n(t)\rangle = \mathbb{U}(t)|n(0)\rangle. \tag{2.22}$$

Equation (2.19) describes the evolution of an eigenstate $|n(0)\rangle$. It is similar to the expression given by Berry (1984a). However, in obtaining his expression, Berry has assumed the evolution of the quantal system to be adiabatic so that $|n(t)\rangle$ represents an instantaneous eigenstate of the Hamiltonian H, while in our case, $|n(t)\rangle$ need not necessarily be an instantaneous eigenstate and in this way, no adiabatic assumption is required.

We would like to stress that the simple expression for the wavefunction in (2.19) is the result of the particular choice of representation specified by (2.9) and (2.11). However the uniqueness of representation of the evolution operator $U(t, 0)$ and (2.6) ensures that the wavefunction in (2.19) is independent of the choice of representation (namely the choice of $\mathbb{U}(t)$ and $\mathbb{R}(t)$). In other words, if we started with another $\mathbb{U}'(t)$ (and correspondingly another $\mathbb{R}'(t)$), the wavefunction thus obtained can, in principle, be reduced to the expression given by (2.19). It should be noticed that the specification made in (2.11) and (2.12) does not induce a unique representation for $\mathbb{U}(t)$. In fact, there is still an infinite number of choices of $\mathbb{U}(t)$ and $\mathbb{R}(t)$ satisfying the above specification. For instance, if we start with another $\bar{\mathbb{U}}(t)$, different from $\mathbb{U}(t)$, so that the corresponding operator $\bar{\mathbb{R}}(t)$ is given by

$$[\bar{\mathbb{R}}(t)]_{m,n} = [\bar{\mathbb{U}}^+(t)U(t,0)]_{m,n} = \delta_{m,n}\exp(-\mathrm{i}\bar{\vartheta}_n(t)) \tag{2.23}$$

with

$$\bar{\vartheta}_n(t) \neq \vartheta_n(t) \tag{2.24}$$

and

$$\bar{\vartheta}_n(0) = 0 \tag{2.25}$$

then $\bar{\mathbb{U}}(t)$ can be related to $\mathbb{U}(t)$ by a unitary transformation:

$$\bar{\mathbb{U}}(t) = \mathbb{U}(t)\mathscr{R}(t) \tag{2.26}$$

in which $\mathscr{R}(t)$ is represented by

$$[\mathscr{R}(t)]_{m,n} = \delta_{m,n} \exp(\mathrm{i}\phi_n(t)) \tag{2.27}$$

with

$$\phi_n(t) = \vartheta_n(t) - \bar{\vartheta}_n(t). \tag{2.28}$$

The wavefunction in the new representation $(\bar{\mathbb{U}}(t), \bar{\mathbb{R}}(t))$, will then be given by

$$|\bar{\Phi}(t)\rangle = \exp(\mathrm{i}\bar{\gamma}_{\mathrm{D}}(t)) \exp(\mathrm{i}\bar{\gamma}_{\mathrm{B}}(t))|\bar{n}(t)\rangle \tag{2.29}$$

where

$$\bar{\gamma}_{\mathrm{D}}(t) = -\frac{1}{\hbar}\int_0^t \langle\Phi(u)|H|\Phi(u)\rangle\,\mathrm{d}u = \gamma_{\mathrm{D}}(t) \tag{2.30}$$

$$\bar{\gamma}_{\mathrm{B}}(t) = \mathrm{i}\int_0^t \langle\bar{n}(u)|\dot{\bar{n}}(u)\rangle\,\mathrm{d}u \tag{2.31}$$

and

$$|\bar{n}(t)\rangle = \bar{\mathbb{U}}(t)|n(0)\rangle. \tag{2.32}$$

We have placed a bar over each symbol to indicate that $\bar{\mathbb{U}}(t)$ is used instead of $\mathbb{U}(t)$.

Employing the notation†

$$A_{m,n} = \mathrm{i}\langle m(0)|\mathbb{U}^{+}(t)\dot{\mathbb{U}}(t)|n(0)\rangle \tag{2.33}$$

the phase factor $\bar{\gamma}_{\mathrm{B}}(t)$ appearing in (2.31) then becomes

$$\bar{\gamma}_{\mathrm{B}}(t) = \int_0^t \bar{A}_{n,n}(u)\,\mathrm{d}u. \tag{2.34}$$

Using (2.26) and (2.27), it can be easily seen that, under the change of representation from $\mathbb{U}(t)$ to $\bar{\mathbb{U}}(t)$, we have the following transformations:

$$|n(t)\rangle \to |\bar{n}(t)\rangle = \exp(\mathrm{i}\phi_n(t))|n(t)\rangle \tag{2.35}$$

$$A_{m,n}(t) \to \bar{A}_{m,n}(t) = \exp[\mathrm{i}(\phi_n(t) - \phi_m(t))][A_{m,n}(t) - \dot{\phi}_n(t)\delta_{m,n}] \tag{2.36}$$

$$\gamma_{\mathrm{B}}(t) \to \bar{\gamma}_{\mathrm{B}}(t) = \gamma_{\mathrm{B}}(t) - \phi_n(t). \tag{2.37}$$

Resetting all these results into (2.29), we arrive at

$$\begin{aligned} |\bar{\Phi}(t)\rangle &= \exp(\mathrm{i}\gamma_{\mathrm{D}}(t)) \exp\{\mathrm{i}[\gamma_{\mathrm{B}}(t) - \phi_n(t)]\} \exp(\mathrm{i}\phi_n(t))|n(t)\rangle \\ &= |\Phi(t)\rangle. \end{aligned} \tag{2.38}$$

Thus the wavefunction can be expressed equivalently by different representation of $\mathbb{U}(t)$ within the specification given by (2.11) and (2.12).

† $A_{m,n}$ plays the role of a gauge potential here.

Now we investigate the evolution of the above wavefunction when it undergoes a cyclic evolution. In other words, we study the evolution of the wavefunction while the quantal system transports round a circuit in the projective Hilbert space, as Aharonov and Anandan (1987) have considered. The above condition can be represented by

$$|\Phi(\tau)\rangle = \exp(\mathrm{i}\gamma)|\Phi(0)\rangle \tag{2.39}$$

for some real γ and τ. Here τ denotes the time whence the quantal system returns to its initial state in the projective Hilbert space, and γ is the total phase change resulting from such an evolution. Now, in view of (2.19), we must have

$$|n(\tau)\rangle = \exp(\mathrm{i}\alpha)|n(0)\rangle \tag{2.40}$$

for some real α. It should be noticed that the value of α depends on the choice of $\mathbb{U}(t)$. In fact, the quantity $\exp(\mathrm{i}\alpha)$ is just the eigenvalue of the operator $\mathbb{U}(\tau)$ with respect to the eigenstate $|n(0)\rangle$. Substituting (2.40) back into (2.29), the wavefunction at time $t = \tau$ will become

$$|\Phi(\tau)\rangle = \exp(\mathrm{i}\gamma_{\mathrm{D}}(\tau))\exp(\mathrm{i}\gamma_{\mathrm{T}}(\tau))|n(0)\rangle \tag{2.41}$$

where

$$\gamma_{\mathrm{T}}(\tau) = \gamma_{\mathrm{B}}(\tau) + \alpha. \tag{2.42}$$

So in view of (2.39), the total phase change γ will be composed of two parts:

$$\gamma = \gamma_{\mathrm{D}}(\tau) + \gamma_{\mathrm{T}}(\tau). \tag{2.43}$$

The first part $\gamma_{\mathrm{D}}(\tau)$ is the usual dynamical phase, while the other part, $\gamma_{\mathrm{T}}(\tau)$, is the topological phase. Due to the fact that the wavefunction is independent of the choice of $\mathbb{U}(t)$, the topological phase $\gamma_{\mathrm{T}}(\tau)$ should also be independent of the choice of $\mathbb{U}(t)$. In fact, it can be easily seen that, with the change of representation from $\mathbb{U}(t)$ to $\bar{\mathbb{U}}(t)$ in (2.26), $\gamma_{\mathrm{B}}(\tau)$ and α transform in the following way:

$$\gamma_{\mathrm{B}}(\tau) \to \bar{\gamma}_{\mathrm{B}}(\tau) = \gamma_{\mathrm{B}}(\tau) - \phi_n(\tau) \tag{2.44}$$

$$\alpha \to \bar{\alpha} = \alpha + \phi_n(\tau). \tag{2.45}$$

Therefore any change in the phase $\gamma_{\mathrm{B}}(\tau)$ due to the change in the representation of $\mathbb{U}(t)$ is compensated by a corresponding change in α.

3. Specific choice of representation

We should note that expression (2.41) represents our general result that an eigenstate, once transported round a closed circuit in the projective Hilbert space, will acquire a topological phase, apart from the usual dynamical phase. These two phases are independent of any particular choice of $\mathbb{U}(t)$. In obtaining these results, we notice that no adiabatic assumption is needed. Besides, the Hamiltonian need not necessarily be time dependent. Furthermore, no underlying parameter space is required to describe the cyclic evolution of the wavefunction. The existence of the toplogical phase merely reflects the geometric property of the physical motion of the quantal system itself. Therefore it becomes possible to associate the toplogical phase with non-trivial cyclic evolution of an isolated system, as pointed out by Anandan (1988).

In the following, we shall make two particular choices of $\mathbb{U}(t)$, and hence $\mathbb{R}(t)$, by which the result made by Berry (1984a) and Anandan (1988) can be regenerated.

(*a*) Firstly, in view of (2.40) and (2.45), the phase angle α can always be transformed away by a particular (but not unique, of course) choice for $\mathbb{U}(t)$ and $\mathbb{R}(t)$ so that we have

$$\alpha = 0 \tag{3.1}$$

and hence

$$|n(\tau)\rangle = \mathbb{U}(\tau)|n(0)\rangle = |n(0)\rangle. \tag{3.2}$$

Then from (2.21) and (2.42), the toplogical phase will be given by

$$\gamma_{\mathrm{T}}(\tau) = \mathrm{i}\int_0^\tau \langle n(0)|\mathbb{U}^+(t)\dot{\mathbb{U}}(t)|n(0)\rangle\, \mathrm{d}t. \tag{3.3}$$

Denoting

$$\varphi = \mathbb{U}(t)|n(0)\rangle \tag{3.4}$$

as a state vector in the Hilbert space, (3.2) then implies that this state vector φ will describe a closed cycle in the Hilbert space while the quantal system is transported round a closed circuit in the projective Hilbert space. The topological phase in (3.3) can thus be written as a closed path integral in the Hilbert space:

$$\gamma_{\mathrm{T}}(\tau) = \mathrm{i}\oint_C \varphi^* \cdot \mathrm{d}\varphi \tag{3.5}$$

with C denoting the trajectory of the state vector φ in the Hilbert space. Now if we consider that the evolution of the state vector φ is induced by a change of parameter $\mathbf{R}$ in a parameter space, then we can represent φ as a function of $\mathbf{R}$:

$$\varphi \equiv \varphi(\mathbf{R}) \equiv |n(t)\rangle \equiv |n(\mathbf{R})\rangle. \tag{3.6}$$

Equation (3.2) thus implies that the state vector $\varphi(\mathbf{R})$ is single valued and is going round a closed cycle in the parameter space. This condition is precisely what Berry (1984) required in his derivation. In the parameter space, the topological phase can be written as a closed path integral:

$$\gamma_{\mathrm{T}}(\tau) = \mathrm{i}\oint_{\bar{C}} n(\mathbf{R})|\nabla_{\mathbf{R}} n(\mathbf{R})\rangle \cdot d\mathbf{R} \tag{3.7}$$

in which $\bar{C}$ is a closed circuit generated by the state vector $|n(\mathbf{R})\rangle$ in the parameter space. If we now employ the adiabatic assumption that the wavefunction $|\Phi(t)\rangle$ is an instantaneous eigenstate of the Hamiltonian H at any time, then in view of (2.2), the evolution operator $U(t, 0)$ will satisfy the following relation:

$$HU(t,0)|n(0)\rangle = E_n(t)\, U(t,0)|n(0)\rangle. \tag{3.8}$$

The evolution operator $U(t, 0)$ here is expressible as product of two unitary operators $\mathbb{U}(t)$ and $\mathbb{R}(t)$ specified by conditions (3.1) and (3.2). Since $\mathbb{R}(t)$ is diagonal in the base $\{|n(0)\rangle\}$, then from (3.8) we see that $\mathbb{U}(t)$ will also satisfy the relation (in addition to condition (3.1))

$$H\mathbb{U}(t)|n(0)\rangle = E_n(t)\mathbb{U}(t)|n(0)\rangle. \tag{3.9}$$

Therefore the state vector $|n(t)\rangle = \mathbb{U}(t)|n(0)\rangle$ is again an instantaneous eigenstate of the Hamiltonian H, and the result represented in (3.7) is exactly the result obtained by Berry. Now, as pointed out in his work, the integral (3.7) is independent of the choice of base of the eigenstates $\{|n(t)\rangle\}$ (i.e. the choice of $\mathbb{U}(t)$). This is, of course, an elucidation of the fact that the wavefunction $|\Phi(t)\rangle$ is independent of the choice of $\mathbb{U}(t)$.

(*b*) Now let us consider another choice of $\mathbb{U}(t)$ given by

$$\langle n(0)|\mathbb{U}^{+}(t)\dot{\mathbb{U}}(t)|n(0)\rangle = 0. \tag{3.10}$$

This choice of $\mathbb{U}(t)$ has been employed by a number of authors (Anandan and Stodolsky 1987, Berry 1987, Anandan 1988) and it implies that the state vector $\mathbb{U}(t)|n(0)\rangle$ is parallel-transported along a closed circuit in the projective Hilbert space. Now in view of expression (3.10), $\gamma_{\mathrm{B}}(\tau)$ will be identically zero. From (2.42), the toplogical phase is then simply given by

$$\gamma_{\mathrm{T}}(\tau) = \alpha \tag{3.11}$$

with α satisfying the eigenequation

$$\mathbb{U}(\tau)|n(0)\rangle = \exp(\mathrm{i}\alpha)|n(0)\rangle. \tag{3.12}$$

This is precisely the result obtained by Anandan (1988). As (3.12) stands, α is not represented by a closed form expression. However, the topological phase $\gamma_{\mathrm{T}}(\tau)$ is independent of the choice of $\mathbb{U}(t)$. Therefore the topological phase α appearing in (3.11) and (3.12) should be identical with the expression shown in (3.5). In this way, we need not necessarily solve the eigenequation (3.12) to obtain α. Instead, (3.5) (or equivalently (3.6)) provides another expression for finding α, the Berry phase.

4. Conclusion

Starting from the evolution equation, we have derived the wavefunction describing the evolution of an eigenstate (2.19), which is shown to be independent of the choice of $\mathbb{U}(t)$. If we consider a quantal system undergoing a cyclic evolution (2.39), as considered by Aharonov and Anandan (1987), the eigenstate will acquire a toplogical phase as well as the usual dynamical phase. These two phases are independent of the choice of $\mathbb{U}(t)$. In our derivation, no adiabatic assumption is required; besides, the Hamiltonian describing the quantal system need not necessarily be time dependent. Therefore our result is valid for any non-adiabatic evolution of an eigenstate. Equation (3.5) gives a general expression for the toplogical phase. Under condition (3.2), this phase becomes the non-adiabatic analogue of Berry's result.

We have used the evolution operator method to derive Berry's phase as indicated by (3.5). Under conditions (3.1) and (3.10), (3.5) respectively translates to Berry's and Anandan's results, which are obtained from different approaches. Such a consequence is just an elucidation of the independence of Berry's phase on the choice of $\mathbb{U}(t)$.

Finally we would like to stress that the topological phase is only a manifestation of the physical motion of a quantal system under a cyclic evolution. Any evolution of an eigenstate, whether adiabatic or not, when transported round the same closed circuit in the projective Hilbert space, will give rise to the same toplogical phase.

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